

# Bergmann–Thomson Energy-Momentum Complex for Solutions More General than the Kerr–Schild Class

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**Abstract** In a very well-known paper, Virbhadra’s research group proved that the Weinberg, Papapetrou, Landau and Lifshitz, and Einstein energy-momentum complexes “coincide” for all metrics of Kerr–Schild class. A few years later, Virbhadra clarified that this “coincidence” in fact holds for metrics more general than the Kerr–Schild class. In the present paper, this study is extended for the Bergmann–Thomson complex and it is proved that this complex also “coincides” with those complexes for a more general than the Kerr–Schild class metric.

**Keywords** Bergmann–Thomson complex · Kerr–Schild class metrics · Energy-momentum problem

## 1 Introduction

Virbhadra and his collaborators (notably, Professor Nathan Rosen of the EPR paradox, Einstein–Rosen bridge, and Einstein–Rosen gravitational waves fame) resurrected the subject of energy-momentum in general relativity [1–13]. They proved that energy-momentum complexes “coincide” and give reasonable results for a particular space-time. They found this to be true for some well-known and physically significant space-times. Further, Aguirregabiria, Chamorro and Virbhadra [14] observed that the energy-momentum complexes of Weinberg, Papapetrou, Landau and Lifshitz (LL), and Einstein all “coincide” for any metric of Kerr–Schild class if calculations are accomplished in Kerr–Schild Cartesian coordinates. A few years later, Virbhadra [15] noted that this “coincidence” is in fact true for space-times more general than the Kerr–Schild class. This paper by Virbhadra triggered a lot of fascinating publications in this subject in international journals and the list is growing exponentially. It is really impossible to discuss most of them in this paper. In the following paragraph, we are able to narrate only some of the results.

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We studied the energy of the Kerr–Newman metric, Bianchi Type I universes, Schwarzschild black hole in a magnetic universe, and nonstatic spherically symmetric metrics [16–20]. Vagenas investigated energy of a radiating charged particle in Einstein’s universe and a dyadosphere of a Reissner–Nordström black hole [21, 22]. He also studied energy distribution in  $(2 + 1)$  dimensional space-times [23–25]. Gad [26] obtained energy density associated with solutions exhibiting directional singularities. Sharif [27–29] computed energy distributions for a regular black hole space-time, the Godel universe, and the Weyl metric. Vargas [30] calculated the energy of the universe in tele-parallel gravity.

Aydogdu and his colleagues studied energy of the universe in Bianchi type-I and II models, Reboucas–Tiomno–Korotkii–Obukhov, Godel, and some other space-times [31–34]. Aygun and his collaborators computed energy and momentum distributions in Szekeres type I and II space-times and the Marder space-time [35]. Halpern [36, 37] obtained energy associated with black plane and Taub cosmological solutions. Further, Salti and his collaborators have accomplished gigantic amount of work in this field of research [38–41]. They studied energy and momentum problem for the Schwarzschild de Sitter space-time, Reissner–Nordström anti-de Sitter black holes, closed universes, a charged wormhole, and for several other interesting cases. For a comprehensive review on the energy-momentum problem in general relativity and some recent important results on this topic, see the Ph.D. thesis of the present author [42].

We discussed in the first paragraph of this section that Virbhadra [15] showed that the energy-momentum complexes of Weinberg, Papapetrou, Landau and Lifshitz, and Einstein “coincide” for a more general metric than the Kerr–Schild class, subject to the condition that calculations are performed in Kerr–Schild Cartesian coordinates. In a recent paper published in Foundation of Physics Letters, we [43] proved that the Bergmann–Thomson complex furnishes the same result for the Kerr–Newman black hole metric as obtained by Aguirregabiria et al. [14] about a decade ago. As a natural flow of research curiosity, results obtained by Virbhadra [15] now tempts us to investigate whether or not the Bergmann–Thomson complex also yields the same result for metrics more general than the Kerr–Schild class or for at least the Kerr–Schild class. Like in our previous papers, we use geometrized units and follow the convention that Latin and Greek indices respectively take values 0 to 3 and 1 to 3.

## 2 Energy-Momentum Complexes of Weinberg, Papapetrou, Landau and Lifshitz, and Einstein

A renowned particle physicist and Nobel laureate Steven Weinberg proposed an energy-momentum complex (see in [42]). This is now termed the *Weinberg energy-momentum complex* and is expressed by the equation

$$\mathcal{W}^{ik} = \left( \frac{1}{16\pi} \right) \omega^{qik}{}_{,q}. \quad (1)$$

In the above equation,

$$\omega^{qik} = \frac{\partial h^r{}_r}{\partial x_q} \eta^{ik} + \frac{\partial h^ri}{\partial x^r} \eta^{qk} + \frac{\partial h^{qk}}{\partial x_i} - \frac{\partial h^r{}_r}{\partial x_i} \eta^{qk} - \frac{\partial h^{rq}}{\partial x^r} \eta^{ik} - \frac{\partial h^{ik}}{\partial x_q} \quad (2)$$

where

$$h_{ik} = g_{ik} - \eta_{ik}. \quad (3)$$

It must be remembered that the indices on  $\partial/\partial x_i$  and  $h_{ik}$  are lowered/raised with aid of  $\eta$ 's, where  $\eta_{ik}$  is the Minkowski metric in 3 + 1 space-time dimensions such that  $\eta^{ab} = \text{diag}(1, -1, -1, -1)$ .

Equation (2) shows that the pseudotensor  $\omega^{qik}$  is antisymmetric in its first two indices, i.e.

$$\omega^{qik} = -\omega^{iqk}. \tag{4}$$

This antisymmetry property helps applying Gauss's theorem and computing energy and momentum inside a closed surface. The Weinberg complex  $\mathcal{W}^{ik}$  is symmetric. The energy and momentum components can be computed using the following formula:

$$\mathbb{P}^k = \iiint \mathcal{W}^{k0} dx^1 dx^2 dx^3. \tag{5}$$

If we use Gauss's theorem in the above equation, we get

$$\mathbb{P}^k = \left(\frac{1}{16\pi}\right) \iint \omega^{\gamma 0k} n_\gamma d\Omega. \tag{6}$$

$d\Omega$  is the infinitesimal surface element and  $n_\gamma$  is the unit normal vector to that surface. In (5),  $\mathcal{W}^{00}$  and  $\mathcal{W}^{\alpha 0}$  are respectively the energy and momentum density components. It was indicated in the Introduction that the Greek indices run from 1 to 3.

A leading Greek relativist Achilleas Papapetrou also obtained a symmetric energy-momentum complex, now known as the *Papapetrou complex* (see in [42]). This is given by the following expression:

$$\mathfrak{B}^{ik} = \left(\frac{1}{16\pi}\right) \mathcal{A}^{ikqr}{}_{,qr}. \tag{7}$$

$\mathcal{A}^{ikqr}$  in the above equation is

$$\mathcal{A}^{ikqr} = \sqrt{-g}(g^{ik}\eta^{qr} + g^{qr}\eta^{ik} - g^{iq}\eta^{kr} - g^{qk}\eta^{ir}), \tag{8}$$

where

$$\eta^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

$\mathfrak{B}^{00}$  and  $\mathfrak{B}^{\alpha 0}$  represent the energy and energy current density components.

If the metric under study were time-independent, then we have an advantage that we can apply Gauss's theorem to calculate energy and momentum components

$$\mathbb{P}^k = \left(\frac{1}{16\pi}\right) \iint \mathcal{A}^{k0\gamma\beta}{}_{,\beta} n_\gamma d\Omega. \tag{10}$$

One of the greatest Russian theoretical physicist and Nobel laureate Landau and his collaborator Lifshitz discovered a symmetric energy-momentum complex (see in [42]). This is called the *Landau and Lifshitz energy-momentum complex* and is given by the following cute mathematical expression:

$$\mathcal{L}^{ik} = \left(\frac{1}{16\pi}\right) \mathcal{A}^{ikqr}{}_{,qr}, \tag{11}$$

with

$$\Lambda^{ikqr} = -g(g^{ik}g^{qr} - g^{iq}g^{kr}). \tag{12}$$

$\mathcal{L}^{00}$  and  $\mathcal{L}^{\alpha 0}$  furnish the energy and momentum density components associated with a given metric. The energy and momentum are

$$\mathbb{P}^k = \iiint \mathcal{L}^{k0} dx^1 dx^2 dx^3. \tag{13}$$

A straightforward application of Gauss’s theorem to the integral on the right hand side of the above equation yields:

$$\mathbb{P}^k = \left(\frac{1}{16\pi}\right) \iiint \Lambda^{k0\gamma q}{}_{,q} n_\gamma d\Omega. \tag{14}$$

We now discuss Einstein’s complex. Einstein was in fact the first to obtain an energy-momentum complex. However, this complex was not symmetric like the aforesaid three we discussed and therefore cannot be used for defining angular momentum in general relativity. This complex is

$$\mathcal{E}_i{}^k = \left(\frac{1}{16\pi}\right) \mathcal{H}_i{}^{kq}{}_{,q}. \tag{15}$$

$\mathcal{H}_i{}^{kl}$  in the above equation is

$$\mathcal{H}_i{}^{kq} = -\mathcal{H}_i{}^{qk} = \frac{g_{is}}{\sqrt{-g}} [-g(g^{ks}g^{qr} - g^{qs}g^{kr})]_{,r}. \tag{16}$$

The energy and momentum components are

$$\mathbb{P}_k = \iiint \mathcal{E}_k{}^0 dx^1 dx^2 dx^3. \tag{17}$$

This equation gives

$$\mathbb{P}_k = \left(\frac{1}{16\pi}\right) \iiint \mathcal{H}_k{}^{0\gamma} n_\gamma d\Omega. \tag{18}$$

### 3 Kerr–Schild Class Space-Times

Let us denote a scalar field by  $\mathcal{S}$  and a vector field by  $V_i$ . Then a Kerr–Schild class metric is well-known to be defined by the metric  $g_{ab}$  as follows:

$$g_{ab} = \eta_{ab} - \mathcal{S} V_a V_b, \tag{19}$$

where  $\eta_{ab}$  is the Minkowski metric as defined in (9) and the vector field  $V_a$  satisfies the following conditions in the Minkowski space-time:

$$\begin{aligned} \eta^{qr} V_q V_r &= 0 && \text{(null condition),} \\ \eta^{qr} V_{a,q} V_r &= 0 && \text{(geodesic condition),} \\ (V_{q,r} + V_{r,q}) V^q{}_{,s} \eta^{rs} - (V^q{}_{,q})^2 &= 0 && \text{(shear-free condition).} \end{aligned} \tag{20}$$

Virbhadra [15] mentioned a few important examples of the Kerr–Schild class space-times:

- Schwarzschild
- Reissner–Nordström
- Kerr
- Kerr–Newman
- Vaidya
- Bonnor–Vaidya
- Vaidya–Patel
- Dybney et al.

For a comprehensive discussion of these space-times, see references in [15]. Aguirregabiria, Chamorro and Virbhadra [14] discovered a marvelous result that the energy-momentum complexes of Weinberg, Papapetrou, Landau and Lifshitz and Einstein “coincide” for any space-time of the Kerr–Schild class. They established the following extremely important relationship among energy-momentum complexes:

$$\begin{aligned} \mathcal{W}^{jk} &= \mathfrak{B}^{jk} = \mathcal{L}^{jk} = \left(\frac{1}{16\pi}\right) \mathbb{U}^{jkr{s}}, \\ \mathcal{E}_i{}^k &= \eta_{iq} \mathcal{L}^{qk}, \end{aligned} \tag{21}$$

with

$$\mathbb{U}^{jkr{s}} = \mathcal{S}(\eta^{ik} V^r V^s - \eta^{ir} V^k V^s + \eta^{rs} V^i V^k - \eta^{ks} V^i V^r). \tag{22}$$

These results shook the prevailing notion that different energy-momentum complexes will meaninglessly give different results for a given metric. Using the above results, they [14] further obtained general expressions for the energy, momentum and angular momentum for any metric of the Kerr–Schild class.

It seems that Aguirregabiria, Chamorro and Virbhadra [14] did not notice whether or not all the three conditions (null, geodesic and shear-free) were used for obtaining (21) with (22). Thanks to Virbhadra [15] that three years later he noticed and reported that only null and geodesic conditions (not the shear-free) were used to derive the relationship given in (21). Thus, this relationship is true for space-times more general than the Kerr–Schild class, because the shear-free condition is not demanded for this derivation.

#### 4 Bergmann–Thomson Complex for a Class of Space-Times More General than the Kerr–Schild Class

About more than fifty years ago, Bergmann and Thomson [44] obtained a new energy-momentum complex:

$$\mathcal{B}^{jk} = \frac{1}{16\pi} \mathbf{B}^{jkq}{}_{,q}, \tag{23}$$

where

$$\mathcal{B}^{jkq} = g^{jr} \mathcal{V}_r{}^{kq} \tag{24}$$

with

$$\mathcal{V}_r{}^{kq} = -\mathcal{V}_r{}^{qk} = \frac{g_{rs}}{\sqrt{-g}} [-g(g^{ks} g^{qp} - g^{qs} g^{kp})]_{,p}. \tag{25}$$

Similar to the four energy-momentum complexes discussed in the last section,  $\mathbf{B}^{jk}$  does not transform as a tensor under a general coordinate transformation.  $\mathcal{B}^{00}$  is the energy density and  $\mathcal{B}^{\alpha 0}$  is the momentum density components.

The energy and momentum components  $\mathbb{P}^k$  are given by

$$\mathbb{P}^k = \int \int \int \mathcal{B}^{k0} dx^1 dx^2 dx^3 = \left( \frac{1}{16\pi} \right) \int \int \mathbf{B}^{k0\gamma} n_\gamma d\Omega. \tag{26}$$

We [43] computed energy and momentum distributions in the Kerr–Newman space-time using the Bergmann–Thomson complex and to a great wonder it came to be the same as obtained by Aguirregabiria, Chamorro and Virbhadra [14] in formulations of Weinberg, Papapetrou, Landau and Lifshitz, and Einstein. Our result encouraged us to investigate further whether the Bergmann–Thomson complex “coincides” with other complexes for any Kerr–Schild class space-times or for a more general class than this.

Let us examine all complexes discussed in Sect. 2 and the Bergmann–Thomson complex meticulously. It is straightforward to prove and is also known (for instance, see in [14]) that for the metric expressed by (19), one has

$$g = -1. \tag{27}$$

It must be remembered that none of the three conditions in (20) is required to obtain the above equation. Only (19) is enough to derive (27). Now (11) and (23) with (27) results

$$\mathcal{L}^{ik} = \mathcal{B}^{ik}. \tag{28}$$

Thus, using (21) and (28) and Virbhadra’s result [15] that only null and geodesic conditions are required to derive (21), we arrive at the following result.

For any space-time more general than the Kerr–Schild class (i.e. space-time described by (19) satisfying only the null and geodesic conditions of (20)), one gets

$$\begin{aligned} \mathcal{W}^{ik} &= \mathfrak{B}^{ik} = \mathcal{L}^{ik} = \mathcal{B}^{ik} = \left( \frac{1}{16\pi} \right) \mathbb{U}^{ikrs}{}_{,rs}, \\ \mathcal{E}_i{}^k &= \eta_{iq} \mathcal{L}^{qk}, \end{aligned} \tag{29}$$

with

$$\mathbb{U}^{ikrs} = \mathcal{S}(\eta^{ik} V^r V^s - \eta^{ir} V^k V^s + \eta^{rs} V^i V^k - \eta^{ks} V^i V^r). \tag{30}$$

### 5 Summary

The concept of energy-momentum distribution in general relativity has not been taken seriously mainly for the following reasons:

- (a) Due to the liberty granted by the divergence relation of an energy-momentum complex, many complexes have been proposed and many more can be discovered. There is no unique definition.
- (b) These complexes are pseudotensors (non-tensors under general coordinate transformations). Therefore, an energy-momentum complex can in principle give different results in different coordinates for any space-time. For instance, Møller investigated the

Schwarzschild metric in Einstein’s prescription. One would be really shocked that the total energy diverges (i.e.  $-\infty$ ) in spherical polar coordinates, whereas it is the expected Schwarzschild mass  $M$  if computations are accomplished in quasi-Cartesian coordinates (refer to [42] for more information).

- (c) Misner, Thorne, and Wheeler [45] expressed that to look for a local gravitational energy-momentum is looking for the right answer to the wrong question; however, for spherical systems the gravitational potential energy is correct and meaningful.

Because of these reasons, this subject remained neglected for a very long period of time until under the leadership of Virbhadra this subject was re-animated. We write a few important points about the recent development.

- (i) Virbhadra showed for several space-times that different energy-momentum complexes give the same and reasonable results (see in [1–8].)
- (ii) A leading relativist *Bondi* [46] expressed his viewpoint: “In relativity, a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found”.
- (iii) A renowned astrophysicist and Nobel laureate *S. Chandrasekhar* showed interest in the energy-momentum complexes [47].
- (iv) Aguirregabiria, Chamorro and Virbhadra [14] showed that the energy-momentum complexes of Weinberg, Papapetrou, Landau and Lifshitz, and Einstein give the same results for any Kerr–Schild class metric.
- (v) Rosen (an eminent collaborator of *Albert Einstein*) and Virbhadra and then again Virbhadra (see in [9] and [1–8]) showed that different energy momentum complexes give the same and reasonable results for the Einstein–Rosen space-time, which is though not of Kerr–Schild class. Clearly, the “coincidence” of different complexes is not confined to the Kerr–Schild class space-times.
- (vi) Virbhadra [15] clarified that the “coincidence” of Weinberg, Papapetrou, Landau and Lifshitz, and Einstein is true for a class of space-times more general than the Kerr–Schild class.
- (vii) The invaluable contributions of Virbhadra’s research team enlivened this subject and now many researchers from different countries started working in this field.
- (viii) In this paper, we (the present author) extended the work of Virbhadra [15] to the case of the Bergmann–Thomson complex.

Despite these fascinating successes, the energy-momentum distribution in a curved space-time is far from settled and much more painstaking efforts are still warranted. This continues to be a very ‘hot’ topic of research in Einstein’s general relativity.

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